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2005 J. Phys. A: Math. Gen. 38 L549

(http://iopscience.iop.org/0305-4470/38/33/L01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) L549–L556 doi:10.1088/0305-4470/38/33/L01

LETTER TO THE EDITOR

Spatial correlations of the 1D KPZ surface on a flat substrate

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Received 20 April 2005, in final form 28 June 2005
Published 3 August 2005
Online at stacks.iop.org/JPhysA/38/L549

Abstract
We study the spatial correlations of the one-dimensional KPZ surface for the flat initial condition. It is shown that the multi-point joint distribution for the height is given by a Fredholm determinant, with its kernel in the scaling limit explicitly obtained. This may also describe the dynamics of the largest eigenvalue in the GOE Dyson's Brownian motion model. Our analysis is based on a reformulation of the determinantal Green's function for the totally ASEP in terms of a vicious walk problem.

PACS numbers: 05.40.−a, 81.10.Aj, 02.50.Ey, 68.55.Ac

Surface growth has been an important subject of physics both from practical and fundamental aspects. While good control of it is crucial in recent atom-scale technology, a rich variety of interesting surface patterns has attracted much attention of theoretical studies [1, 2]. It is also important from the point of view of nonequilibrium statistical mechanics.

It is in general difficult to obtain detailed information about the properties of surfaces by analytical methods. However, in one spatial dimension some surface growth models are known to be exactly solvable. They are very special in many respects, but give us much insight into understanding the properties of surfaces in nature. The Kardar–Parisi–Zhang (KPZ) equation, introduced in [3], is one of the minimal models in the theory of surface growth which have both nonlinear and noise effects. Many models were shown to belong to the same universality class as the KPZ equation, i.e., the KPZ universality class [4]. But the analysis was mainly restricted to the exponents for some time.

A next breakthrough comes from an observation that some surface growth models, in particular the polynuclear growth (PNG) model, are related to the combinatorial problem of Young tableaux [5]. In [6, 7], the height fluctuation of the surface in the KPZ universality class was shown to be equivalent to that of the largest eigenvalue of random matrices [8].

One of the important results is that the height fluctuation at one point strongly depends on boundary and initial conditions [6, 9, 10]. For instance, droplet growth, in which the surface starts from a seed and grows into a droplet shape, is related to the Gaussian unitary ensemble (GUE); the model on a flat substrate is related to the Gaussian orthogonal ensemble (GOE).

More recently, spatial correlations of the surface, which have information about how rough the surface is, have been calculated. For the droplet growth, it was shown to be equivalent to the dynamics of the largest eigenvalue of the GUE Dyson's Brownian motion model [11–14]. The PNG model with external sources and that in half-space have also been studied [15–17]. All these results have been obtained by introducing the multi-layer version of the PNG model [12, 13]. A relationship between the topology of multi-layers and the universality classes in random matrix theory has also been discussed [18].

The spatial correlations for the flat case are important for several reasons. This case is suitable for studying how an initially straight surface without fluctuation grows into a rough one. Many simulations have been performed for the flat case. In addition, there is a more theoretical interest. It is possible to define the multi-layer PNG model for the flat case and the fluctuations of the multi-layers at one point are shown to be the same as those of the largest eigenvalues of the GOE [19]. Combined with the spatial homogeneity of the flat case, this suggests that the flat case is related to the GOE Dyson's Brownian motion model [11], which is equivalent to the $\beta = 1$ case of the Calogero model [20, 21]. Since this is not a free fermion model, the analysis of the flat case might well be qualitatively more difficult than for other cases. Note that the constraints of the multi-layers are complicated and have not allowed us to study the multi-point fluctuation.

In this letter, we tackle this problem by a related but different method, i.e., by utilizing a formula for the one-dimensional totally asymmetric simple exclusion process (TASEP) on an infinite lattice. There are several versions of the update rules for TASEP. Here we consider the continuous time version, which is the most well studied. In TASEP, each site is either occupied by a particle or empty. In an infinitesimal time duration dt each particle tries to hop to the right nearest site with probability dt. The hopping is not allowed if the target site is already occupied by another particle. This is a very simple model but exhibits much interesting behaviour [22–25].

We can interpret the TASEP dynamics as a kind of surface growth model if we replace each occupied site with a slope of -45° and each empty site with a slope of 45°; see figure 1. Hopping of a particle corresponds to local surface growth in the surface growth picture. This surface growth model is known to belong to the KPZ universality class [4]. The flat initial condition corresponds to the initial surface configuration in which ╱ and ╲ occur alternately. We study the spatial correlations of this model.

If we describe the stochastic dynamics of TASEP in terms of a master equation, it turns out that the transition rate matrix can be written as a kind of non-Hermitian spin chain [25, 26]. In the language of spin chains, the TASEP is a kind of XXZ model and hence is *not* a free fermion model. If one applies the Bethe ansatz method to TASEP, the S-matrix is not just -1. But the Bethe ansatz analysis is useful to understand the temporal properties of the ASEP [26, 27]. For instance, the Bethe ansatz method allows us to construct Green's function in the form of determinant [28]. Let $G(x_1, x_2, \ldots, x_N; t|y_1, y_2, \ldots, y_N; 0)$ denote the probability that the N particles are on sites $x_1, x_2, \ldots, x_N$ $(x_N < \cdots < x_1)$ at time t under the condition that they are on sites $y_1, y_2, \ldots, y_N$ $(y_N < \cdots < y_1)$ at time 0. Then $G(x_1, x_2, \ldots, x_N; t|y_1, y_2, \ldots, y_N; 0)$ is given by

$$G(x_1, x_2, \ldots, x_N; t|y_1, y_2, \ldots, y_N; 0) = \det[F_{k-j}(x_{N-k+1} - y_{N-j+1}; t)]_{j,k=1}^{N}. \quad (1)$$

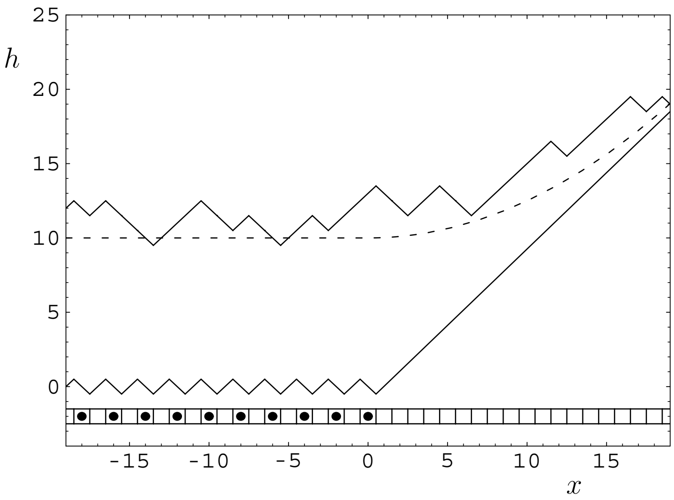

Figure 1. The initial configuration of the ASEP and the corresponding surface. A surface after some time is also shown with its asymptotic position (the dotted line).

Here the function $F_n(x;t)$ is defined by

$$F_n(x;t) = \mathrm{e}^{-t}\frac{t^x}{x!}\sum_{k=0}^{\infty}\frac{(n)_k}{(x+1)_k}\frac{t^k}{k!} \tag{2}$$

where $(n)_k = n(n+1)\cdots(n+k-1)$. When we fix y_j, the initial configuration of particles, we call this quantity $G(x_1, x_2, \ldots, x_N; t)$ as well. This formula has already been used for studying the fluctuation properties of the TASEP and a discrete version of it [29, 30].

Now, on the other hand, let us consider a weight in the form of products of determinants,

$$\prod_{r=1}^{N-1}\det\left[\phi\left(x_j^r, x_k^{r+1}\right)\right]_{j,k=1}^{r+1}\det\left[\psi_j^{(N)}\left(x_{k+1}^N\right)\right]_{j,k=0}^{N-1}, \tag{3}$$

on x_j^r $(r = 1, \ldots, N,\ j = 1, \ldots, r)$. We put the condition $x_1^2 < x_2^2, x_1^3 < x_2^3 < x_3^3, \ldots$ and the convention $\phi\left(x_{r+1}^r, x_k^{r+1}\right) = 1$ for $r = 1, \ldots, N-1,\ k = 1, \ldots, r+1$. Here the functions $\psi_j^{(r)}(x)$ and $\phi(x_1, x_2)$ are defined by

$$\psi_j^{(r)}(x) = (-1)^{r-1-j}F_{-r+1+j}(x - y_{j+1}; t) \tag{4}$$

$(j = 0, \ldots, r-1)$ and $\phi(x_1, x_2) = 0\ (x_1 > x_2), -1\ (x_1 \leqslant x_2)$. Let $\mathbb{P}$ denote the corresponding measure. Then we have

$$G(x_1, \ldots, x_N; t) = \mathbb{P}\left[x_1^r = x_r\ (r = 1, 2, \ldots, N)\right]. \tag{5}$$

This is one of the main results of this letter and is proved as follows. First, one shows

$$\prod_{r=1}^{N-1}\sum_{x_{r+1}^{r+1} > \cdots > x_2^{r+1} (> x_1^{r+1})}\prod_{r=1}^{N-1}\det\left[\phi\left(x_j^r, x_k^{r+1}\right)\right]_{j,k=1}^{r+1} f = \prod_{r=1}^{N-1}\prod_{l=2}^{r+1}\sum_{x_l^{r+1} = x_{l-1}^r}^{\infty} f \tag{6}$$

with f being an arbitrary antisymmetric function of x_j^N. Here the summation on the left-hand side is over all possible configurations of x_j^r $(r = 1, \ldots, N,\ j = 2, \ldots, r)$ with x_1^r being fixed to x_r $(r = 1, \ldots, N)$. This can be shown by considering when the determinant vanishes, using the antisymmetry of the function f and the mathematical induction. Then using a property of the function $F_n(x;t)$, $F_{n+1}(x;t) = \sum_{y=x}^{\infty} F_n(y;t)$, one arrives at (5).

If we interpret r as a time coordinate and x_j^r as the position of the jth walker at r, the weight in (3) can be interpreted as a kind of vicious walk problem with a peculiar structure that each particle is added at each time step. The formula (5) allows us to obtain the multi-point information of our surface model from the trajectory of the first particle, x_1^r $(r = 1, 2, \ldots)$, in the vicious walk problem. The vicious walk problem was introduced in [31] and is regarded as a sort of free fermion model [32]. Hence (5) explicitly states that one can study the dynamics of the TASEP in terms of free fermions even though the transition rate matrix does not look like a free fermion model.

For each r let us define another set of functions, $\varphi_j^{(r)}(x)$ $(j = 0, \ldots, r-1)$, which is a polynomial of order $r - 1 - j$, by the condition that they are orthonormal to $\psi_j^{(r)}(x)$,

$$\sum_{x=y_r}^{\infty} \varphi_j^{(r)}(x)\psi_k^{(r)}(x) = \delta_{jk}. \tag{7}$$

We also define $\phi_{r_1,r_2}(x_1, x_2)$ by

$$\phi_{r_1,r_2}(x_1, x_2) = (-1)^{r_2-r_1} \int_{\Gamma_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{z^{x_1-x_2-1}}{(1-z)^{r_2-r_1}} \tag{8}$$

for $r_1 < r_2$ and $\phi_{r_1,r_2}(x_1, x_2) = 0$ for $r_1 \geqslant r_2$. Here Γ_0 is a contour enclosing the origin anticlockwise. Note $\phi_{r,r+1}(x_1, x_2) = \phi(x_1, x_2)$.

We can show that our vicious walk problem is determinantal, i.e., the correlation functions of the system are written in the determinant form. For instance, the probability that $x_l^{r_i}$ $(l = 1, 2, \ldots, r_i)$ are occupied by walkers at r_i $(i = 1, 2)$ is proportional to

$$\det\left[K\left(r_{i_1}, x_{l_1}^{r_{i_1}}; r_{i_2}, x_{l_2}^{r_{i_2}}\right)\right]_{i_1,i_2=1,2,l_1=1,\ldots,r_{i_1},l_2=1,\ldots,r_{i_2}}. \tag{9}$$

Here the matrix element is given by

$$\begin{aligned} K(r_1, x_1; r_2, x_2) &= \tilde{K}(r_1, x_1; r_2, x_2) - \phi_{r_1,r_2}(x_1, x_2) \\ &= \sum_{j=0}^{N-1} \psi_j^{(r_1)}(x_1)\varphi_j^{(r_2)}(x_2) - \phi_{r_1,r_2}(x_1, x_2). \end{aligned} \tag{10}$$

We set $\varphi_j^{(r)}(x) = 0$ for $j \geqslant r$ and hence the summation is actually up to $r_2 - 1$. This fact can be proved by following the strategy of [33] but there appear some new features compared to the usual case of a fixed number of walkers [17]. For instance, when $r_1 < r_2$, $\psi_j^{(r)}$ with $j \geqslant r$ are included in the summation of (10).

Then the joint distribution of the first particle in our vicious walk problem, which corresponds to the joint distribution of the height in our original surface growth model, can be described by a Fredholm determinant with the kernel given by the function $K(r_1, x_1; r_2, x_2)$:

$$\begin{aligned} &\mathbb{P}\left[x_1^{r_i} > X_i (i = 1, \ldots, m)\right] \\ &\quad = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{i_1=1}^{m} \sum_{x_1=y_{r_{i_1}}}^{X_1} \cdots \sum_{i_k=1}^{m} \sum_{x_k=y_{r_{i_k}}}^{X_k} \det\left[K\left(r_{i_{l_1}}, x_{l_1}; r_{i_{l_2}}, x_{l_2}\right)\right]_{l_1,l_2=1}^{k}. \end{aligned} \tag{11}$$

Up to here our discussions are for general initial configuration. We believe that there are many applications of the formula (5) to study the temporal properties of TASEP. In this letter, we concentrate on analysing the spatial correlations of the KPZ surface for the flat initial condition, which has not been solved by the usual multi-layer PNG techniques.

We take a special initial condition, $y_{j+1} = -2j\ (j = 0, \ldots, N-1)$. See figure 1. This is not exactly the flat initial condition, but deep inside the negative ($x < 0$) region, the correlations are the same as the flat case because the effect of the boundary is restricted near the origin at finite time. One can also restrict the number of particles to N since the dynamics of a particle cannot affect the dynamics of particles on its right. Let $h(x,t)$ denote the surface height at position x and at time t. The limiting shape is known to be $h(x,t)/t \sim \frac{1}{2}(x \leqslant 0), \frac{1}{2}\big(1+\frac{x^2}{t^2}\big)(0 < x \leqslant t), x/t\,(x \geqslant t)$. We are interested in the fluctuation around this.

For the special choice $y_{j+1} = -2j$, one can find an explicit formula for $\varphi_j^{(r)}(x)$,

$$\varphi_j^{(r)}(x) = \frac{(-1)^{r-1-j}}{2\pi \mathrm{i}} \int_{\Gamma_0} \mathrm{d}z \frac{1+2z}{1+z} \frac{(1+z)^{x+r+j-1}}{z^{r-j}}\, \mathrm{e}^{-zt}. \tag{12}$$

It is not difficult to check the orthonormality relation (7). As a consequence, the kernel has a double contour integral expression,

$$K(r_1,x_1;r_2,x_2) = \int_{\Gamma_{-1}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \int_{\Gamma_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{(1+z)^{x_2+r_2-2}(-w)^{r_1}(1+2z)\,\mathrm{e}^{(w-z)t}}{(1+w)^{x_1+r_1-1}(-z)^{r_2}(w-z)(1+w+z)} - \tilde{\phi}_{r_1,r_2}(x_1,x_2) \tag{13}$$

where Γ_{-1} is a contour enclosing -1 anticlockwise and

$$\tilde{\phi}_{r_1,r_2}(x_1,x_2) = \int_{\Gamma_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} z^{x_2+r_2-x_1-r_1-1}(1-z)^{r_1-r_2}. \tag{14}$$

Now we consider the scaling limit, in which universal properties of the model are expected to appear. First let us consider the positive ($x > 0$) region. In this case, it is convenient to set

$$r_i = t\rho_i = t\rho + 2\sqrt{\rho}\,t^{2/3}\tau_i/d, \tag{15}$$

$$x_i + r_i - 1 = (1-\sqrt{\rho_i})^2 t - (1-\sqrt{\rho})\,d t^{1/3}\xi_i, \tag{16}$$

where $d = \rho^{-1/6}(1-\sqrt{\rho})^{-1/3}$ for $i = 1, 2$, and take the $t \to \infty$ limit with ρ fixed. Note that $0 < \rho < 1/4$ corresponds to looking at the positive ($x > 0$) region. Applying the saddle point analysis to (13), we get the limiting kernel,

$$\mathcal{K}_2(\tau_1,\xi_1;\tau_2,\xi_2) = \tilde{\mathcal{K}}_2(\tau_1,\xi_1;\tau_2,\xi_2) - \Phi_2(\tau_1,\xi_1;\tau_2,\xi_2) \tag{17}$$

where

$$\tilde{\mathcal{K}}_2(\tau_1,\xi_1;\tau_2,\xi_2) = \int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{-\lambda(\tau_1-\tau_2)} \mathrm{Ai}(\xi_1+\lambda)\mathrm{Ai}(\xi_2+\lambda), \tag{18}$$

$$\Phi_2(\tau_1,\xi_1;\tau_2,\xi_2) = \int_{-\infty}^\infty \mathrm{d}\lambda\, \mathrm{e}^{-\lambda(\tau_1-\tau_2)} \mathrm{Ai}(\xi_1+\lambda)\mathrm{Ai}(\xi_2+\lambda) \tag{19}$$

for $\tau_1 < \tau_2$ and $\Phi_2(\tau_1,\xi_1;\tau_2,\xi_2) = 0$ for $\tau_1 \geqslant \tau_2$. This is the kernel for the Airy process [12, 13]. Hence we conclude that the fluctuation of surface in the positive ($x > 0$) region is the same as that of the droplet growth. The spatial correlation of the surface is the same as the dynamics of the largest eigenvalue in the GUE Dyson's Brownian motion model.

Next we consider the negative ($x < 0$) region. We want to study the fluctuation of the scaled height,

$$A_1(\tau) = 2t^{-1/3}\Big(t/2 - h\Big(x = \big(\tfrac{1}{2} - 2\rho\big)t - t^{2/3}\tau, t\Big)\Big) \tag{20}$$

for $\rho > 1/4$ as $t \to \infty$. This corresponds to setting

$$r_i = t\rho + t^{2/3}\tau_i/2, \tag{21}$$

$$x_i + 2r_i - 2 = t/2 - t^{1/3}\xi_i/2, \tag{22}$$

in the kernel and take the $t \to \infty$ limit. In the scaling limit, a contribution from the pole at $z = -1 - w$ in (13),

$$\tilde{K}_1(r_1, x_1; r_2, x_2) = \int_{\Gamma_{-1}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{(-w)^{x_2+r_1+r_2-2}\,\mathrm{e}^{(2w+1)t}}{(1+w)^{x_1+r_1+r_2-1}}, \tag{23}$$

turns out to be dominant. Applying the saddle point analysis to (23), we get the limiting kernel,

$$\mathcal{K}_1(\tau_1, \xi_1; \tau_2, \xi_2) = \tilde{\mathcal{K}}_1(\tau_1, \xi_1; \tau_2, \xi_2) - \Phi_1(\tau_1, \xi_1; \tau_2, \xi_2) \tag{24}$$

where

$$\tilde{\mathcal{K}}_1(\tau_1, \xi_1; \tau_2, \xi_2) = \frac{1}{2}\,\mathrm{e}^{(\tau_2-\tau_1)(\xi_1+\xi_2)/4+(\tau_2-\tau_1)^3/12}\mathrm{Ai}\left(\frac{\xi_1+\xi_2}{2} + \frac{(\tau_2-\tau_1)^2}{4}\right), \tag{25}$$

$$\Phi_1(\tau_1, \xi_1; \tau_2, \xi_2) = \frac{1}{\sqrt{8\pi(\tau_2-\tau_1)}} \exp\left[-\frac{(\xi_2-\xi_1)^2}{8(\tau_2-\tau_1)}\right] \tag{26}$$

for $\tau_1 < \tau_2$ and $\Phi_1(\tau_1, \xi_1; \tau_2, \xi_2) = 0$ for $\tau_1 \geqslant \tau_2$. The result is valid for the whole negative ($x < 0$) region. The fact that the multi-point joint distribution of the 1D KPZ surface on a flat substrate is given by the Fredholm determinant of this kernel is another main result of this letter.

We remark that the Fredholm determinant of the kernel with $\tau_1 = \tau_2 (= \tau)$,

$$\mathcal{K}_1(\tau, \xi_1; \tau, \xi_2) = \frac{1}{2}\mathrm{Ai}\left(\frac{\xi_1+\xi_2}{2}\right), \tag{27}$$

gives the one point height fluctuation in the negative ($x < 0$) region and hence should be equivalent to the GOE Tracy–Widom distribution function $F_1(s)$ [34]. Though we do not know how to prove this directly at the moment, the statistics computed from the kernel agrees very well with the GOE values numerically.

The kernels, $\mathcal{K}_1$ and $\mathcal{K}_2$, have a lot of statistical information about the surface. For instance, using the joint distribution at time 0 and τ, we can compute

$$g_n(\tau) = \sqrt{\langle (A_n(\tau) - A_n(0))^2 \rangle / 2} \tag{28}$$

for $n = 1, 2$. Here $A_1(\tau)$ is defined in (20) and $A_2(\tau)$ is the Airy process [12, 13]. It is clear that $\lim_{\tau \downarrow 0} g_n(\tau) = 0$ and $g_1(\tau)$ (respectively $g_2(\tau)$) approaches the standard deviation of the GOE (respectively GUE) largest eigenvalue 1.2680 (respectively 0.9018) as $\tau \to \infty$. In figure 2, we show the comparison of this quantity between our theoretical predictions and Monte Carlo simulation results for the surface growth model. The agreement is very good.

Combined with the conjecture in [19], the fluctuation of the largest eigenvalue in the GOE Dyson's Brownian motion model may also be described by the kernel, $\mathcal{K}_1$. It is desirable to have a better understanding of this connection. It is also an interesting question to see if the joint distribution for the GOE case satisfies some differential equations. They would be helpful to understand the properties of our process $A_1(\tau)$. For $A_2(\tau)$, such differential equations are already known [35, 36].

To summarize, we have shown that Green's function for the TASEP can be interpreted as a vicious walk problem. This demonstrates a hidden free fermionic structure behind the TASEP and opens up a possibility of analysing time-dependent properties of the model in the language of free fermions. By using the formula, we have found an exact expression for the multi-point joint distribution of the KPZ surface for the flat initial condition. It is written

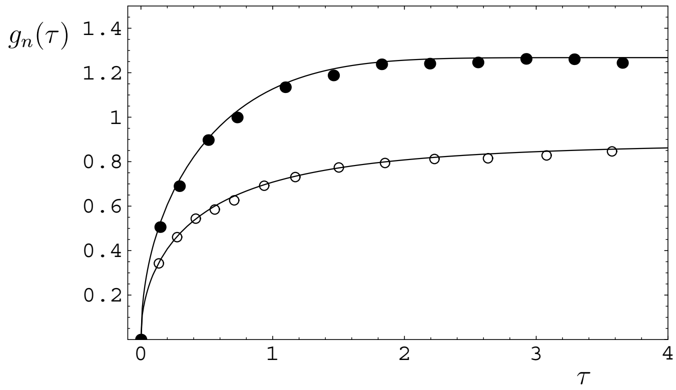

Figure 2. Comparison of the correlation $g_n(\tau)$ $(n = 1, 2)$ computed from the Fredholm expressions (solid lines) and Monte Carlo simulations (circles). The upper ones are for the negative $(x < 0)$ region and the lower ones for the positive $(x > 0)$ region.

in the form of the Fredholm determinant and the kernel in the scaling limit has been explicitly obtained. This is expected to be universal for the KPZ surface on a flat substrate and may have relevance for the study of the GOE Dyson's Brownian motion. More detailed analysis and results for the discrete TASEP will be reported elsewhere [37].

Acknowledgments

The author would like to thank H Spohn, M Katori, H Tanemura, G M Schütz, T Nagao, M Prähofer, A Rákos, P Ferrari and T Imamura for useful discussions and comments. This work is partly supported by the Grant-in-Aid for Young Scientists (B), the Ministry of Education, Culture, Sports, Science and Technology, Japan.

Note added. After the submission of this letter, a direct proof of the fact that the GOE Tracy–Widom distribution is given by the Fredholm determinant with the kernel (27) was given in [38].

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